

Classical and quantum q -deformed physical systems

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Received: date / Revised version: date

Abstract. On the basis of the non-commutative q -calculus, we investigate a q -deformation of the classical Poisson bracket in order to formulate a generalized q -deformed dynamics in the classical regime. The obtained q -deformed Poisson bracket appears invariant under the action of the q -symplectic group of transformations. In this framework we introduce the q -deformed Hamilton's equations and we derive the evolution equation for some simple q -deformed mechanical systems governed by a scalar potential dependent only on the coordinate variable. It appears that the q -deformed Hamiltonian, which is the generator of the equation of motion, is generally not conserved in time but, in correspondence, a new constant of motion is generated. Finally, by following the standard canonical quantization rule, we compare the well known q -deformed Heisenberg algebra with the algebra generated by the q -deformed Poisson bracket.

PACS. 02.45.Gh Noncommutative geometry – 45.20.-d Formalism in classical mechanics – 03.65.-w Quantum mechanics – 02.20.Uw Quantum groups

1 Introduction

Quantum algebra and quantum groups arise as the underlying mathematical structure in several physical phenomena. It has been shown that such a formalism can play an important role in conformal field theory, exact soluble models in statistical physics [1] and in a wide range of applications, from cosmic strings and black holes to solid state physics problems [2, 3, 4].

Many physical applications have been investigated on the basis of the q -deformation of the Heisenberg algebra [5, 6, 7, 8, 9]. For instance, q -deformed Schrödinger equations have been proposed in the literature [10, 11] and applications to the study of q -deformed version of the hydrogen atom and of the quantum harmonic oscillator [12, 13, 14] have been presented. In Ref. [15] the Weyl-Heisenberg algebra has been studied in the framework of the Fock-Bargmann representation allowing a rigorous treatment of the squeezed states, lattice quantum mechanics and Bloch functions.

The theory of the q -deformed harmonic oscillator, based on the construction of $SU_q(2)$ algebra of q -deformed commutation or anticommutation relations between creation and annihilation operators [16, 17, 18], has opened the possibility of studying intermediate q -boson and q -fermion statistical behavior [19, 20]. A kinetic approach of this problem, within a semiclassical treatment, was presented in Ref. [21]. Moreover, it has recently lead to the

application of q -calculus in the construction of generalized statistical mechanics where nonextensivity properties can arise from the q -deformed theory [22]. Along these lines, a generalized thermostatistics based on the formalism of q -calculus has been formulated in Ref. [23], whereas in Ref. [24] a q -deformed entropy was applied to study a gas of q -deformed bosons.

In the recent past, some tentative approaches to construct a q -deformed version of classical mechanics have been investigated. The primary motivation for such a study is to understand the origin of the q -deformed Heisenberg algebra which forms the basis of the deformed quantum mechanics. In other words, we can ask: how does one introduce a q -deformed algebraic structure for the quantum plane coordinates, such that, after canonical quantization, the q -deformed Heisenberg algebra follows? This question was investigated, for instance, in Ref. [25] where the quasi-classical limit of the q -oscillator has been discussed and a q -deformed version of the Poisson bracket (PB) was derived in terms of variables of the quantum plane. A related question is how one can describe the dynamical evolution of a classical object existing in such a q -deformed quantum plane. In Ref. [26] the author has presented a tentative formulation to construct q -deformed classical mechanics based on the introduction of a q -Lagrangian and a q -Hamiltonian where the equations of motion are derived from the q -deformed analog of the Euler-Lagrange equation. A similar formulation was presented in Ref. [27] by introducing a deformed phase-space based

on the elliptic algebra with different deformed parameters for the space coordinates and the momenta.

Finally, among the many motivations for the study of a q -deformed generalization of classical mechanics, it is important to mention the relevance of symmetries in physics [28]. In this respect, quantum groups give a new symmetric aspect for the resolution of classical problems. Along these lines, Ref. [29] deals with the investigation of a set of Poisson algebra structures obtained from the elliptic quantum algebra. It has been shown that the resulting Poisson structure contains the q -deformed Virasoro algebra which plays a central role in the resolution of several integrable systems both in quantum mechanics and statistical mechanics.

Recently, in Ref. [30], a possible definition of q -deformed PB has been derived by requiring that it be invariant under the action of the q -deformed symplectic group $\text{Sp}_q(1)$, in analogy with the classical (undeformed) case, where PB are invariant under the action of the symplectic group $\text{Sp}(1)$. In this paper we start by considering a q -deformed version of PB previously introduced but following another approach, and in some sense, a more systematic derivation instead of the one adopted in Ref. [30].

In the commutative classical mechanics, the PB between two functions $f(x, p)$ and $g(x, p)$ can be defined through the contraction of the corresponding Hamiltonian fields X_f and X_g with the canonical symplectic form $\omega = dx \wedge dp$. In the same manner, we can define a q -deformed version of PB between two q -functions $\hat{f}(\hat{x}, \hat{p})$ and $\hat{g}(\hat{x}, \hat{p})$, defined on the noncommutative q -plane, through the contraction of the corresponding q -Hamiltonian fields $\hat{X}_{\hat{f}}$ and $\hat{X}_{\hat{g}}$ with the q -deformed canonical symplectic form $\hat{\omega} = d\hat{x} \wedge d\hat{p}$ (throughout this paper we denote by a hat the elements belonging to the q -algebra to distinguish them from the element belonging to the ordinary commutative algebra). We then attempt to formulate, by means of the q -deformed Hamilton's equations, a q -deformed classical mechanics describing the time evolution of a mechanical system. Finally, in analogy with standard canonical quantization method, where the (undeformed) PB between canonically conjugate variables x and p are replaced by the (undeformed) commutator of the corresponding quantum operators \bar{x} and \bar{p} , we discuss the scenario of a possible canonical quantization in the q -deformed framework, putting in correspondence the algebra generated by the q -deformed PB between \hat{x} and \hat{p} with the well known algebra generated by the q -deformed commutator of the corresponding q -deformed operators \tilde{x} and \tilde{p} .

Our paper is organized as follows. After a brief review of the derivation of the standard PB in the formalism of the exterior calculus, presented in Sec. 2, we introduce, in Sec. 3, the q -commutative phase-space and recall the definition of its q -calculus. On the basis of the previous sections, we will be able to obtain the most original result of our paper: the introduction of a q -deformed PB in Sec. 4 and the formulation of a possible q -deformed mechanics by means of q -Hamilton's equations is presented in Sec. 5. In Sec. 6, starting from the q -deformed PB we explain a

possible derivation of the q -deformed Heisenberg algebra. Finally the conclusions are presented in Sec. 7.

2 Poisson bracket in the commutative phase space

We begin by recalling briefly the derivation of the standard PB in the formalism of the exterior calculus, referring to the relevant literature for the details [31, 32].

Let us consider the real plane \mathbb{R}^2 generated by the commutative coordinates $x^1 \equiv x$ and $x^2 \equiv p$ and introduce the associative algebra $\mathcal{A} = \text{Fun}(\mathbb{R}^2)$ of the functions on \mathbb{R}^2 freely generated by the elements x and p .

The tangent space $T\mathcal{A}$ on \mathcal{A} is generated by the vectors $\partial_1 \equiv \partial_x = \partial/\partial x$ and $\partial_2 \equiv \partial_p = \partial/\partial p$ which are linear operators i.e., $\partial_i(\lambda f + \mu g) = \lambda \partial_i f + \mu \partial_i g$ and satisfying the Leibniz rule $\partial_i(x^j f) = \delta_i^j f + x^j \partial_i f$ with $i, j = 1, 2$, where $f(x, p)$ and $g(x, p)$ are smooth functions on \mathcal{A} and λ, μ are ordinary commutative numbers (C -numbers). Any vector v can be spanned on the base of ∂_i as $v = f^1 \partial_x + f^2 \partial_p$, where $f^i \in \mathcal{A}$.

In the same way we introduce the cotangent space (one-forms) $T^*\mathcal{A}$, generated by the elements dx and dp . Any one-form ω can be spanned on this base as $\omega = dx g_1 + dp g_2$, where $g_i \in \mathcal{A}$.

Higher order forms are constructed by means of the differential operator d which takes k -forms into $(k+1)$ -forms. In particular starting from a function $f \in \mathcal{A}$ (0-form), its differential is a 1-form

$$df \equiv dx \partial_x f + dp \partial_p f, \quad (1)$$

and, starting from a 1-form $\omega = dx g_1 + dp g_2$ we obtain the 2-form

$$d\omega \equiv dx \wedge dp (\partial_p g_2 - \partial_x g_1), \quad (2)$$

where we have introduced the exterior product between two 1-forms, that is linear $(\lambda \omega_1 + \mu \omega_2) \wedge \omega_3 = \lambda (\omega_1 \wedge \omega_3) + \mu (\omega_2 \wedge \omega_3)$, associative $(\omega_1 \wedge \omega_2) \wedge \omega_3 = \omega_1 \wedge (\omega_2 \wedge \omega_3)$ and skew-symmetric $\omega_1 \wedge \omega_2 = -\omega_2 \wedge \omega_1$.

The d operator fulfills the following main properties: $d(\lambda \omega + \mu \omega') = \lambda d\omega + \mu d\omega'$ (linearity); $d(fg) = df g + f dg$ (Leibniz rule); $d\lambda = 0$; $d(d\omega) = 0$; and $d(\omega^{(k)} \wedge \omega^{(p)}) = d\omega^{(k)} \wedge \omega^{(p)} + (-1)^k \omega^{(k)} \wedge d\omega^{(p)}$ where k and p are the degree of $\omega^{(k)}$ and $\omega^{(p)}$, respectively.

Finally, we introduce the contraction operator $i_v(\omega)$, in the axiomatic way, through its main propriety listed below,

$$\begin{aligned} i_v(f) &= 0, \\ i_{\partial_i}(dx^j) &= \delta_i^j, \\ i_{(\lambda f \partial_i + \mu g \partial_j)}(\omega) &= \lambda f i_{\partial_i}(\omega) + \mu g i_{\partial_j}(\omega), \\ i_v(\lambda dx^i f + \mu dx^j g) &= \lambda i_v(dx^i) f + \mu i_v(dx^j) g, \\ i_{\partial_i}(dx^1 \wedge dx^2) &= \delta_i^1 dx^2 - \delta_i^2 dx^1. \end{aligned} \quad (3)$$

In order to derive the Poisson bracket and their algebra within this formalism we begin by introducing the symplectic form

$$\omega = dx \wedge dp, \quad (4)$$

and define the Hamiltonian vector field X_f , associated with a function $f \in \mathcal{A}$, through the relation

$$i_{X_f}(\omega) = df. \quad (5)$$

From Eqs. (4) and (5), taking into account the properties of the contraction operator (3), it follows that, for any function f , the corresponding Hamiltonian vector field X_f assumes the expression

$$X_f \equiv \partial_p f \partial_x - \partial_x f \partial_p. \quad (6)$$

As a consequence, we can write the Poisson bracket between the two functions $f, g \in \mathcal{A}$ through the relation

$$\{f, g\} \equiv i_{X_g}(df) = i_{X_g} i_{X_f}(\omega). \quad (7)$$

It is easy to show, accounting for Eq. (6), that Eq. (7) can be written in the usual form

$$\{f, g\} = \partial_x f \partial_p g - \partial_p f \partial_x g. \quad (8)$$

We recall the main properties of the PB defined through Eq. (7) which are: bi-linearity $\{\lambda f + \mu g, h\} = \lambda \{f, h\} + \mu \{g, h\}$ and $\{f, \lambda g + \mu h\} = \lambda \{f, g\} + \mu \{f, h\}$, skew symmetry $\{f, g\} = -\{g, f\}$ and the Jacobi identity $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$.

We remark that Eq. (8) can also be expressed in the form

$$\{f, g\} = \partial_i f J^{ij} \partial_j g, \quad (9)$$

(repeated indexes are summed over), where we have introduced the structure functions

$$J^{ij} = \{x^i, x^j\}, \quad (10)$$

which can be arranged in 2×2 matrix J . Taking into account the expression of the symplectic form (4), we easily recognize that J is the symplectic unity, with entries $J^{ij} = \epsilon^{ij}$, where $\epsilon^{12} = -\epsilon^{21} = 1$.

It is easy to show that Eq. (9) is invariant in form under a symplectic transformation $\text{Sp}(1)$ on the plane \mathbb{R}^2

$$\{f, g\} \rightarrow \{f', g'\} = \partial'_i f' J^{ij} \partial'_j g', \quad (11)$$

with $f' = f(x', p')$ and $g' = g(x', p')$ and

$$x^i \rightarrow x'^i = x^j T_j^i, \quad (12)$$

where T_j^i , the entries of a matrix $T \in \text{Sp}(1)$, satisfies the symplectic relation

$$T_m^i J^{mn} T_n^j = J^{ij}. \quad (13)$$

3 q -commutative differential calculus

In order to generalize the PB in the framework of a q -deformed theory, we review the basic properties of the q -commutative differential calculus.

The real quantum plane $\widehat{\mathbb{R}}^2$ is generated by the q -commutative element $\hat{x}^1 \equiv \hat{x}$ and $\hat{x}^2 \equiv \hat{p}$, obeying the relation [33]

$$\hat{p} \hat{x} = q \hat{x} \hat{p}, \quad (14)$$

which is invariant under the action of $\text{Gl}_q(2)$ transformations and q is the real deformation parameter. We denote by $\widehat{\mathcal{A}} = \text{Fun}(\widehat{\mathbb{R}}^2)$ the associative algebras freely generated by the elements \hat{x} and \hat{p} .

In analogy with the commutative case, we define the q -tangent space $T\widehat{\mathcal{A}}$ [34], generated by the q -derivatives $\hat{\partial}_1 \equiv \hat{\partial}_x$ and $\hat{\partial}_2 \equiv \hat{\partial}_p$, whose action on the generators \hat{x}^i is defined as [5]

$$\hat{\partial}_i \hat{x}^j = \delta_i^j. \quad (15)$$

They are linear operators satisfying $\hat{\partial}_i(\lambda \hat{f} + \mu \hat{g}) = \lambda \hat{\partial}_i \hat{f} + \mu \hat{\partial}_i \hat{g}$ which fulfill the q -Leibniz rule

$$\begin{aligned} \hat{\partial}_p \hat{p} &= 1 + q^2 \hat{p} \hat{\partial}_p + (q^2 - 1) \hat{x} \hat{\partial}_x, \\ \hat{\partial}_p \hat{x} &= q \hat{x} \hat{\partial}_p, \\ \hat{\partial}_x \hat{p} &= q \hat{p} \hat{\partial}_x, \\ \hat{\partial}_x \hat{x} &= 1 + q^2 \hat{x} \hat{\partial}_x, \end{aligned} \quad (16)$$

leading to the q -commutative derivative

$$\hat{\partial}_p \hat{\partial}_x = q^{-1} \hat{\partial}_x \hat{\partial}_p. \quad (17)$$

Let us now introduce the q -cotangent space $T^*\widehat{\mathcal{A}}$, generated by the elements $d\hat{x}$ and $d\hat{p}$. Any q -deformed 1-form $\hat{\omega}$ can be spanned on this basis as $\hat{\omega} = d\hat{x} \hat{g}_1 + d\hat{p} \hat{g}_2$ where $\hat{g}_i \in \widehat{\mathcal{A}}$. We observe that, $d\hat{x}^i \hat{f} \neq \hat{f} d\hat{x}^i$, due to the q -commutative structure of the q -calculus. Nevertheless, accounting for the following relations

$$\begin{aligned} \hat{p} d\hat{p} &= q^2 d\hat{p} \hat{p}, \\ \hat{x} d\hat{p} &= q d\hat{p} \hat{x}, \\ \hat{p} d\hat{x} &= q d\hat{x} \hat{p} + (q^2 - 1) d\hat{p} \hat{x}, \\ \hat{x} d\hat{x} &= q^2 d\hat{x} \hat{x}, \end{aligned} \quad (18)$$

any q -deformed 1-form is well defined and admits a unique, left or right, expansion: $\hat{\omega} = \hat{f}_i^L d\hat{x}^i = d\hat{x}^i \hat{f}_i^R$ where the quantities \hat{f}_i^R and \hat{f}_i^L can be obtained each from the other by means of Eqs. (18).

Through the action of the operator d we can construct higher order q -deformed forms. In particular, the differential of a q -function $\hat{f} \in \widehat{\mathcal{A}}$ is given by

$$d\hat{f} \equiv d\hat{x} (\hat{\partial}_x \hat{f})^R + d\hat{p} (\hat{\partial}_p \hat{f})^R, \quad (19)$$

which, by means of Eqs. (18), can be written equivalently in the (left) form

$$d\hat{f} \equiv (\hat{\partial}_x \hat{f})^L d\hat{x} + (\hat{\partial}_p \hat{f})^L d\hat{p}. \quad (20)$$

We recall that in the framework of the q -calculus, the operator d fulfills the same formal properties as in standard calculus. The elements $d\hat{x}$ and $d\hat{p}$ satisfy the relations

$$\begin{aligned} d\hat{p} \wedge d\hat{p} &= 0, \\ d\hat{p} \wedge d\hat{x} &= -q^{-1} d\hat{x} \wedge d\hat{p}, \\ d\hat{x} \wedge d\hat{x} &= 0, \end{aligned} \quad (21)$$

where, the exterior product is still linear and associative. Finally, we introduce the contraction operator $i_v(\omega)$ between q -deformed vectors and q -deformed forms, in the axiomatic way, through its main properties listed below

$$\begin{aligned} i_{\hat{v}}(\hat{f}) &= 0, \\ i_{\hat{\partial}_i}(d\hat{x}^j) &= \delta_i^j, \\ i_{(\lambda \hat{f} \hat{\partial}_i + \mu \hat{g} \hat{\partial}_j)}(\hat{\omega}) &= \lambda \hat{f} i_{\hat{\partial}_i}(\hat{\omega}) + \mu \hat{g} i_{\hat{\partial}_j}(\hat{\omega}), \\ i_{\hat{v}}(\lambda d\hat{x}^i \hat{f} + \mu d\hat{x}^j \hat{g}) &= \lambda i_{\hat{v}}(d\hat{x}^i) \hat{f} + \mu i_{\hat{v}}(d\hat{x}^j) \hat{g}, \\ i_{\hat{\partial}_k}(d\hat{x}^1 \wedge d\hat{x}^2) &= \delta_k^1 d\hat{x}^2 - q^{-1} \delta_k^2 d\hat{x}^1, \end{aligned} \quad (22)$$

which are formally identical to Eq. (3) with the difference that now, the ordering is important. A rigorous derivation of these properties can be found in Ref. [34].

A realization of the above q -algebra and its q -calculus can be accomplished by the replacements [35]

$$\hat{x} \rightarrow x, \quad (23)$$

$$\hat{p} \rightarrow p D_x, \quad (24)$$

$$\hat{\partial}_x \rightarrow \mathcal{D}_x, \quad (25)$$

$$\hat{\partial}_p \rightarrow \mathcal{D}_p D_x, \quad (26)$$

where

$$D_x = q^{x \partial_x} \Rightarrow D_x f(x, p) = f(qx, p), \quad (27)$$

is the dilatation operator along the x direction (reducing to the identity operator for $q \rightarrow 1$), whereas

$$\mathcal{D}_x = \frac{q^{2x \partial_x} - 1}{(q^2 - 1)x}, \quad (28)$$

$$\mathcal{D}_p = \frac{q^{2p \partial_p} - 1}{(q^2 - 1)p}, \quad (29)$$

are the Jackson derivatives (JD) with respect to x and p [36]. Their action on an arbitrary function $f(x, p)$ is

$$\mathcal{D}_x f(x, p) = \frac{f(q^2 x, p) - f(x, p)}{(q^2 - 1)x}, \quad (30)$$

$$\mathcal{D}_p f(x, p) = \frac{f(x, q^2 p) - f(x, p)}{(q^2 - 1)p}, \quad (31)$$

which reduce to the ordinary derivatives when q goes to unity. Therefore, as a consequence of the non-commutative structure of the q -plane, in this realization the \hat{x} coordinate becomes a C -number and its derivative is the JD whereas the \hat{p} coordinate and its derivative are realized in terms of the dilatation operator and JD.

4 Poisson bracket in the q -commutative phase space

On the basis of the q -commutative differential calculus, in this section we derive the expression for the q -deformed PB by following, in analogy, the same formal steps used in

the classical derivation reviewed in Sec. 2. To begin with, let us introduce the q -deformed symplectic form

$$\hat{\omega} = q^{-1/2} d\hat{x} \wedge d\hat{p}, \quad (32)$$

and we define the q -Hamiltonian field \hat{X}_f , associated with the function $\hat{f} \in \hat{\mathcal{A}}$, through the relation

$$i_{\hat{X}_f} \hat{\omega} = d\hat{f}. \quad (33)$$

According to q -calculus, the expression for \hat{X}_f is given by

$$\hat{X}_f = q^{1/2} (\hat{\partial}_p \hat{f})^L \hat{\partial}_x - q^{-1/2} (\hat{\partial}_x \hat{f})^L \hat{\partial}_p, \quad (34)$$

which reduces to the standard Hamiltonian field (5) in the $q \rightarrow 1$ limit.

We introduce the q -Poisson bracket between the q -deformed functions \hat{f} and \hat{g} by means of the relation,

$$\{\hat{f}, \hat{g}\}_q \equiv i_{\hat{X}_{\hat{g}}}(d\hat{f}) = i_{\hat{X}_{\hat{g}}} i_{\hat{X}_{\hat{f}}}(\hat{\omega}). \quad (35)$$

Accounting for Eq. (20) and the properties (22) we obtain the general expression for the q -PB

$$\{\hat{f}, \hat{g}\}_q \equiv q^{1/2} (\hat{\partial}_p \hat{g})^L (\hat{\partial}_x \hat{f})^R - q^{-1/2} (\hat{\partial}_x \hat{g})^L (\hat{\partial}_p \hat{f})^R. \quad (36)$$

Properties of the q -PB (35) are consequences of the properties of the q -deformed contraction operator (22). In particular they are bi-linear,

$$\{\lambda \hat{f} + \mu \hat{g}, \hat{h}\}_q = \lambda \{\hat{f}, \hat{h}\}_q + \mu \{\hat{g}, \hat{h}\}_q, \quad (37)$$

$$\{\hat{f}, \lambda \hat{g} + \mu \hat{h}\}_q = \lambda \{\hat{f}, \hat{g}\}_q + \mu \{\hat{f}, \hat{h}\}_q, \quad (38)$$

but in general they are not skew-symmetric. This can be seen, for instance, if we consider the q -deformed generator functions which can be constructed by setting $\hat{f} \equiv \hat{x}$ and $\hat{g} \equiv \hat{p}$. The canonical q -Hamiltonian fields are respectively

$$\hat{X}_x = -q^{-1/2} \hat{\partial}_p, \quad (39)$$

$$\hat{X}_p = q^{1/2} \hat{\partial}_x, \quad (40)$$

and, after observing that, according to Eq. (15), $(\hat{\partial}_i \hat{x}^j)^L \equiv (\hat{\partial}_i \hat{x}^j)^R$, from Eq. (36) we immediately derive the q -deformed structure functions for the q -Poisson algebra

$$\{\hat{x}^i, \hat{x}^j\}_q = q^{1/2} \hat{\partial}_x \hat{x}^i \hat{\partial}_p \hat{x}^j - q^{-1/2} \hat{\partial}_p \hat{x}^i \hat{\partial}_x \hat{x}^j. \quad (41)$$

Their explicit values are obtained as follows

$$\begin{aligned} \{\hat{x}, \hat{x}\}_q &= \{\hat{p}, \hat{p}\}_q = 0, \\ \{\hat{x}, \hat{p}\}_q &= q^{1/2}, \\ \{\hat{p}, \hat{x}\}_q &= -q^{-1/2}, \end{aligned} \quad (42)$$

so that we easily obtain

$$\{\hat{x}, \hat{p}\}_q = -q \{\hat{p}, \hat{x}\}_q. \quad (43)$$

Finally, the Jacobi identity is trivially satisfied for the q -structure functions

$$\begin{aligned} & \left\{ \hat{x}^i, \left\{ \hat{x}^j, \hat{x}^k \right\}_q \right\}_q + \left\{ \hat{x}^j, \left\{ \hat{x}^k, \hat{x}^i \right\}_q \right\}_q \\ & + \left\{ \hat{x}^k, \left\{ \hat{x}^i, \hat{x}^j \right\}_q \right\}_q = 0. \end{aligned} \quad (44)$$

In Ref. [30] a q -deformed Poisson bracket has been obtained by requiring that, in analogy with the classical case, where standard PB is invariant with respect to a symplectic transformation, the q -PB should be invariant with respect to a q -symplectic transformation. As a result we obtained a q -PB formally equivalent to Eq. (36) but with the replacement of $q \rightarrow 1/q^2$. It is also easy to see that the q -PB given in Eq. (36) is preserved under the action of a q -symplectic transformation. In order to show such a property, we recall that, in the fundamental representation of $\text{Sp}_q(1)$, any element \hat{T} is given by a 2×2 matrix, with entries \hat{T}_i^j , satisfying the following equation [33]

$$\hat{T}_r^i C_q^{rs} \hat{T}_s^j = C_q^{ij}, \quad (45)$$

where $C_q^{ij} = \epsilon^{ij} q^{-\epsilon^{ij}}$. Observing that Eq. (36) can be expressed as

$$\{\hat{f}, \hat{g}\}_q = (\hat{\partial}_i \hat{g})^L \mathcal{J}_q^{ij} (\hat{\partial}_j \hat{f})^R, \quad (46)$$

where $\mathcal{J}_q = -C_{q'}$, with $q' = \sqrt{q}$, according to the property (45), it is easy to see that under the action of quantum group $\text{Sp}_{q'}(1)$, Eq. (46) transforms as

$$\{\hat{f}, \hat{g}\}_q \rightarrow \{\hat{f}', \hat{g}'\}_q = (\hat{\partial}'_i \hat{g}')^L \mathcal{J}_q^{ij} (\hat{\partial}'_j \hat{f}')^R, \quad (47)$$

where $\hat{f}' \equiv \hat{f}(\hat{x}', \hat{p}')$, $\hat{g}' \equiv \hat{g}(\hat{x}', \hat{p}')$, with

$$\hat{x}^i \rightarrow \hat{x}'^i = \hat{x}^j \hat{T}_j^i, \quad (48)$$

and we assumed the commutations between the group elements and the plane elements.

It may be observed that Eq. (45) can be rewritten as

$$\hat{T}_r^i \mathcal{J}_q^{rs} \hat{T}_s^j = \mathcal{J}_q^{ij}, \quad (49)$$

which mimics, in this way, the classical expression (13), where the matrix \mathcal{J}_q plays the role of the symplectic unit J introduced in Eq. (10) and recovered in the $q \rightarrow 1$ limit. The q -deformed generator functions are then related to the entries of the matrix \mathcal{J}_q as

$$\{\hat{x}^i, \hat{x}^j\}_q = \hat{\partial}_r \hat{x}^i \mathcal{J}_q^{rs} \hat{\partial}_s \hat{x}^j, \quad (50)$$

which also shows their invariance under the action of the q -symplectic group $\text{Sp}_{q'}(1)$.

5 q -deformed Hamilton's equations

As a preliminary application of the q -PB derived in the previous section, it is natural to investigate the effect due to the q -commutativity of the coordinates of the phase space on the time evolution of a classical object existing in this space. We postulate that the dynamics in the deformed phase space is described, in analogy with classical mechanics, by means of the following q -deformed evolution equations written in the form

$$\dot{\hat{x}}(t) = \left\{ \hat{x}(t), \hat{H}(\hat{x}, \hat{p}) \right\}_q, \quad (51)$$

$$\dot{\hat{p}}(t) = \left\{ \hat{p}(t), \hat{H}(\hat{x}, \hat{p}) \right\}_q. \quad (52)$$

It is assumed that time enters in the q -generators as a normal parameter. The time derivative, indicated by a dot, means $\dot{\hat{x}} = d\hat{x}/dt$ where dt is a C -number. In this way, the q -commutative algebra of $\dot{\hat{x}}(t)$ and $\dot{\hat{p}}(t)$ and its q -calculus are the same as that of $d\hat{x}(t)$ and $d\hat{p}(t)$. $\hat{H}(\hat{x}, \hat{p})$ is the q -Hamiltonian function which is assumed to not depend explicitly on time.

According to Eq. (36) the evolution Eqs. (51) and (52) can be written down in the form

$$\dot{\hat{x}}(t) = q^{1/2} \left(\hat{\partial}_p \hat{H} \right)^L, \quad (53)$$

$$\dot{\hat{p}}(t) = -q^{1/2} \left(\hat{\partial}_x \hat{H} \right)^L, \quad (54)$$

which are the q -deformed Hamilton's canonical equations.

It can be shown that, as a consequence of Eqs. (51) and (52), the time evolution of any dynamical function $\hat{f}(\hat{x}(t), \hat{p}(t); t)$, depending on the generators $\hat{x}(t)$ and $\hat{p}(t)$, can be described by means of the following evolution equation

$$\frac{d}{dt} \hat{f}(\hat{x}, \hat{p}; t) = \left\{ \hat{f}(\hat{x}, \hat{p}; t), \hat{H}(\hat{x}, \hat{p}) \right\}_q + \frac{\partial}{\partial t} \hat{f}(\hat{x}, \hat{p}; t), \quad (55)$$

where, the last term in the right hand side takes into account the explicit dependence of \hat{f} from t .

In fact, we recall readily that the most general form of a function $\hat{f} \in \hat{\mathcal{A}}$ can be written as a polynomial in the q -variables $\hat{x}(t)$ and $\hat{p}(t)$

$$f(\hat{x}(t), \hat{p}(t); t) = \sum_{n,m} c_{nm}(t) [\hat{x}(t)]^n [\hat{p}(t)]^m, \quad (56)$$

where $c_{nm}(t)$ are C -numbers which may be time dependent and we have assumed the $\hat{x}\hat{p}$ ordering prescription which can be always accomplished by means of Eq. (14). Let us consider the generic term in Eq. (56). Its time derivative becomes

$$\begin{aligned} \frac{d}{dt} (c_{nm} \hat{x}^n \hat{p}^m) &= c_{nm} [n]_q \frac{d\hat{x}}{dt} \hat{x}^{n-1} \hat{p}^m \\ &+ c_{nm} [m]_q q^n \frac{d\hat{p}}{dt} \hat{x}^n \hat{p}^{m-1} + \frac{\partial c_{nm}}{\partial t} \hat{x}^n \hat{p}^m, \end{aligned} \quad (57)$$

where Eq. (18) has been employed and where we have introduced the q -basic number

$$[n]_q = \frac{q^{2n} - 1}{q^2 - 1}. \quad (58)$$

By employing the equations of motion (51) and (52) in the form $d\hat{x}/dt = i_{\hat{X}_H}(d\hat{x})$ and $d\hat{p}/dt = i_{\hat{X}_H}(d\hat{p})$, and accounting for the properties of the operators d and $i_{\hat{v}}$, we obtain

$$\begin{aligned} \frac{d}{dt} (c_{nm} \hat{x}^n \hat{p}^m) &= c_{nm} [n] i_{X_H}(d\hat{x}) \hat{x}^{n-1} \hat{p}^m \\ &\quad + c_{nm} [m] q^n i_{X_H}(d\hat{p}) \hat{x}^n \hat{p}^{m-1} + \frac{\partial c_{nm}}{\partial t} \hat{x}^n \hat{p}^m \\ &= i_{X_H} (c_{nm} ([n] d\hat{x} \hat{x}^{n-1} \hat{p}^m \\ &\quad + [m] q^n d\hat{p} \hat{x}^n \hat{p}^{m-1})) + \frac{\partial}{\partial t} (c_{nm} \hat{x}^n \hat{p}^m) \\ &= i_{X_H} (d(c_{nm} \hat{x}^n \hat{p}^m)) + \frac{\partial}{\partial t} (c_{nm} \hat{x}^n \hat{p}^m) \\ &= \left\{ c_{nm} \hat{x}^n \hat{p}^m, \hat{H}(\hat{p}, \hat{x}) \right\} + \frac{\partial}{\partial t} (c_{nm} \hat{x}^n \hat{p}^m), \quad (59) \end{aligned}$$

which, by linearity, implies Eq. (55).

Finally, we recall that, in standard mechanics, the canonical transformations are a kind of coordinate transformations mixing x and p in a way that preserves the Hamilton structure of the dynamical system. An important property of the canonical transformations is that they preserve the Poisson bracket. Thus, this family of transformations, also called symplectic transformations [31], are realized by the symplectic group transformations. In the q -deformed case, the situation is very similar because, as shown in the previous section, the q -PB are invariant under the action the q -deformed symplectic group. In this sense, the quantum group $\text{Sp}_{q'}(1)$ is realized as a version of q -deformed canonical transformation. It is outside the scope of this work to discuss this important subject of the theory and its implications which will be discussed in a future investigation. In the following we are going to consider some non relativistic systems described by the q -Hamiltonian

$$\hat{H}(\hat{x}, \hat{p}) = \frac{\hat{p}^2}{2m} + \hat{V}(\hat{x}), \quad (60)$$

where m is an ordinary C -number and $\hat{V}(\hat{x})$ is the external potential which we assume to be an arbitrary polynomial in the generator \hat{x} , with C -numbers coefficients. Within this formalism let us discuss some examples.

5.1 The q -free particle

As a first simple example, we choose $\hat{V}(\hat{x}) = 0$, a free particle. The q -Hamiltonian field is readily computed from Eq. (34) and is given by

$$\hat{X}_{H_0} = \frac{\hat{p}}{m_q} \hat{\partial}_x, \quad (61)$$

where $\hat{H}_0 = \hat{p}^2/2m$ and $m_q = 2m q^{3/2}/[2]_q$.

The equations of motion are obtained by employing Eq. (35) and read

$$\dot{\hat{x}} = \frac{\hat{p}}{m_q}, \quad \dot{\hat{p}} = 0. \quad (62)$$

These equations show that the effect of the deformation is to rescale the mass m of the particle as the effective mass m_q . We remark that the q -Hamiltonian of the free particles \hat{H}_0 is a constant of motion of the system. In fact, by using Eq. (55), it can be verified that $\{\hat{H}_0, \hat{H}_0\}_q = 0$ so that the Hamiltonian \hat{H}_0 could represent the energy of the free particle which is conserved in time.

This fact, in the undeformed canonical theory, is merely a consequence of the skew-symmetry of the PB: $\{f, g\} = -\{g, f\}$ which implies $\{f, f\} = 0$. In the q -deformed case, because the q -PB is no longer skew-symmetric, the q -PB evaluated between the same function \hat{f} in general does not vanish. An immediate consequence of this is that, contrary to standard classical mechanics, in the q -deformed theory, the Hamiltonian function in general is not conserved in time. Such a situation has been encountered also in other proposed q -deformed classical systems [27]. Let us investigate the consequence in the next example.

5.2 The q -harmonic oscillator

We consider the harmonic oscillator with a potential

$$\hat{V}_h(\hat{x}) = \frac{1}{2} m \omega^2 \hat{x}^2, \quad (63)$$

where the angular frequency ω is a C -number. The q -Hamiltonian field is evaluated as

$$\hat{X}_{H_h} = \frac{\hat{p}}{m_q} \hat{\partial}_x - m_q \omega_q^2 \hat{x} \hat{\partial}_p, \quad (64)$$

where $\hat{H}_h = \hat{H}_0 + \hat{V}_h$ and $\omega_q = \omega [2]_q / 2 q^2$ while the equations of motion become

$$\dot{\hat{x}} = \frac{\hat{p}}{m_q}, \quad \dot{\hat{p}} = -m_q \omega_q^2 \hat{x}. \quad (65)$$

The effect of the deformation is thus taken into account only through a rescaling of both the mass $m \rightarrow m_q$ and the angular frequency $\omega \rightarrow \omega_q$ of the harmonic oscillator. The time evolution of Hamiltonian \hat{H}_h can be determined by employing Eq. (55) and we obtain

$$\dot{\hat{H}}_h = \left\{ \hat{H}_h, \hat{H}_h \right\}_q = q^{1/2} (q^2 - 1) \omega_q^2 \hat{p} \hat{x}, \quad (66)$$

which vanishes only in the $q \rightarrow 1$ limit. As a consequence, the Hamiltonian \hat{H}_h cannot be associated with the total energy of the conservative system.

Nevertheless, the following function

$$\mathcal{E}_q = \frac{\hat{p}^2}{2m} + \frac{1}{2q^2} m \omega^2 \hat{x}^2, \quad (67)$$

is time conserved, $\dot{\mathcal{E}}_q = 0$, and reduces, in the $q \rightarrow 1$ limit, to the Hamiltonian of the harmonic oscillator. Hence, the non-conservation of \hat{H} is not necessarily fatal for the theory due to the existence of other constants of motions that could be identified with the energy of the system instead of \hat{H} . In other words, in the q -deformed theory, \hat{H} is the generator of the equation of motion which now does not necessarily coincide with the total energy of the system, but in correspondence, a new constant of motion may be generated.

5.3 The general case

The previous result can be extended to the general case by considering a q -deformed mechanical system governed by a potential $\hat{V}(\hat{x})$ defined by a polynomial series of \hat{x}

$$\hat{V}(\hat{x}) = \sum_{n=1} c_n \hat{x}^n, \quad (68)$$

with C -number coefficients. The q -Hamiltonian field is given by

$$\hat{X}_H = \frac{\hat{p}}{m_q} \hat{\partial}_x - q^{-1/2} (\hat{\partial}_x \hat{V})^L \hat{\partial}_p, \quad (69)$$

where $(\hat{\partial}_x \hat{V})^L d\hat{x} = d\hat{x} (\hat{\partial}_x \hat{V})^R$, whereas the equation of motion can be expressed in the form

$$m_q \ddot{\hat{x}} = \hat{F}_q, \quad (70)$$

with $\hat{F}_q = -q^{-1/2} (\hat{\partial}_x \hat{V})^L$ the q -deformed external force. Again, the Hamiltonian is not conserved in time, whereas the function

$$\mathcal{E}_q = \frac{\hat{p}^2}{2m} + \sum_{n=1} d_n \hat{x}^n, \quad (71)$$

with $d_n = c_n q^{4-3n}$, reduces to the Hamiltonian function of the system in the $q \rightarrow 1$ limit and fulfills the relation $\dot{\mathcal{E}}_q = 0$.

6 Canonical quantization

Let us now compare, by means of canonical quantization, the well-known q -deformed Heisenberg algebra with the algebra (42) generated by the q -deformed PB. To start with we recall that the q -deformed Heisenberg algebra reads [6, 7, 8, 10]

$$[\tilde{x}, \tilde{p}]_q = i \tilde{\Lambda}_q, \quad (72)$$

$$\tilde{\Lambda}_q \tilde{x} = q^{-1} \tilde{x} \tilde{\Lambda}_q, \quad \tilde{\Lambda}_q \tilde{p} = q \tilde{p} \tilde{\Lambda}_q, \quad (73)$$

where

$$[\tilde{x}, \tilde{p}]_q = q^{1/2} \tilde{x} \tilde{p} - q^{-1/2} \tilde{p} \tilde{x}, \quad (74)$$

is the q -deformed commutator and we have denoted by a tilde the q -deformed quantum generators \tilde{x} and \tilde{p} . The

extra generator $\tilde{\Lambda}_q$ in Eqs. (72) and (73) plays the role of a dilatator and, in the $q \rightarrow 1$ limit, reduces to the identity operator [7, 37].

As it is known, in the undeformed case, the simplest canonical quantization prescription is given by identify the position variable x with the corresponding multiplicative operator \bar{x} and the canonically conjugate variables p is assumed to be proportional to the space derivative according to the rule

$$x \rightarrow \bar{x}, \quad p \rightarrow \bar{p} \equiv -i \partial_x. \quad (75)$$

At the same time, the undeformed PB is replaced with the undeformed commutator

$$\{x, p\} = 1 \Leftrightarrow [\bar{x}, \bar{p}] = i, \quad (76)$$

where $[\bar{x}, \bar{p}] = \bar{x} \bar{p} - \bar{p} \bar{x}$. In analogy with this scenario, it appears natural to impose, in the q -deformed framework, the following quantization rule on the q -variables

$$\hat{x} \rightarrow \tilde{X}, \quad (77)$$

$$\hat{p} \rightarrow \tilde{P} \equiv -i \hat{\partial}_{\tilde{X}}, \quad (78)$$

and to replace the q -PB with the q -deformed commutator (74).

It should be observed that, contrary to the q -deformed classical theory developed starting from the 2-dimensional q -deformed calculus, the q -deformed quantum mechanics is spanned in the 1-dimensional configuration space. This requires us to consider the 1-dimensional q -deformed calculus generated by the elements \hat{x} , $\hat{\partial}_x$ and $d\hat{x}$ which differs from the corresponding one in 2-dimensions and introduced in Section 3. In particular, the Leibnitz rule now reads [5]

$$\hat{\partial}_x \hat{x} = 1 + q \hat{x} \hat{\partial}_x, \quad (79)$$

which differs from the last of Eq. (16).

Let us observe that, due to Eq. (79), assuming \tilde{X} is a Hermitian quantity, the prescription (78) deals with a non Hermitian definition of momentum. In accordance with current literature, we can define a physical momentum

$$\tilde{p} = \frac{1}{2} (\tilde{P} + \tilde{P}^\dagger), \quad (80)$$

where the Hermitian conjugation of $\hat{\partial}_X$ is defined by

$$\hat{\partial}_X^\dagger = -q^{-1/2} \tilde{\Lambda}_q \hat{\partial}_X, \quad (81)$$

and the unitary operator $\tilde{\Lambda}_q^\dagger = \tilde{\Lambda}_q^{-1}$ takes the expression [7, 37]

$$\tilde{\Lambda}_q = q^{-1/2} [1 + (q-1) \hat{X} \hat{\partial}_X]^{-1}. \quad (82)$$

After the redefinition

$$\tilde{x} = \frac{2q}{1+q} \tilde{X}, \quad (83)$$

it is easy to verify that the definitions (80), (82) and (83) fulfils Eqs. (72) and Eq. (73). Thus, by posing formally

$$\hat{x} \rightarrow \tilde{x}, \quad \hat{p} \rightarrow \tilde{p}, \quad \sqrt{q} \rightarrow i \tilde{\Lambda}_q, \quad (84)$$

we can state the following correspondence

$$\{\hat{x}, \hat{p}\}_q = \sqrt{q} \Leftrightarrow [\tilde{x}, \tilde{p}]_q = i\tilde{\Lambda}_q, \quad (85)$$

among the q -PB between the classical space phase variables and the q -commutator between the corresponding quantum operators.

Finally, by taking the Hermitian conjugate of Eq. (72), accounting for the unitarity of $\tilde{\Lambda}_q$, we obtain

$$[\tilde{p}, \tilde{x}]_q = -i\tilde{\Lambda}_q^{-1}. \quad (86)$$

We can verify that Eqs. (85) and (86) are completely consistent with Eqs. (42), under the correspondence (84). Equivalently, we can verify that the relation

$$\{\hat{x}, \hat{p}\}_q = -\{\hat{x}, \hat{p}\}_q^{-1}, \quad (87)$$

which follows directly from Eqs. (42), is replaced by

$$[\tilde{x}, \tilde{p}]_q = -[\tilde{x}, \tilde{p}]_q^{-1}, \quad (88)$$

as can be verified by a direct calculation.

In light of the above prescriptions, starting from the classical evolution equation (55), it appears natural to postulate the q -deformed Heisenberg equation for an operator \tilde{O}_q as follows

$$\frac{\partial \tilde{O}_q}{\partial t} = i[\tilde{O}_q, \tilde{H}]_q + \frac{\partial \tilde{O}_q}{\partial t}. \quad (89)$$

7 Conclusion

In this paper, we have developed a q -deformed version of the Poisson bracket by generalizing, in the quantum groups framework, the method based on the exterior calculus for the definition of the classical PB. We have derived, in a systematic way, an expression of the q -PB which substantially equivalent to the one recently obtained by us in Ref. [30] by following a different and more consistent method. The two approaches differ in the following sense. In the previous one [30], the expression for the q -PB was conjectured by requiring the invariance of the q -PB under the action of the symplectic group $\text{Sp}_q(1)$, leading to the q -deformation of the phase space, whereas in the present investigation, only the non-commutative q -deformation of the phase space has been imposed and the q -PB has been obtained as a consequence of the q -calculus. It has been shown that the new version of the deformed bracket is still invariant under the action of the q -symplectic group of transformations $\text{Sl}_{q'}(2)$ with $q' = \sqrt{q}$.

We have discussed some properties of the q -deformed PB and we have presented some elementary examples to illustrate how a possible q -deformed classical mechanics can be introduced. It has been shown that, in contrast to the undeformed case, the q -Hamiltonian, which is the generator of the evolution equation, is in general not conserved

in time and cannot be identified with the total energy for conservative systems. However, a suitable function, reducing to the Hamiltonian function in the $q \rightarrow 1$ limit and remaining constant during the evolution of the system, has been obtained for the examples studied. These properties, related to derivation of the q -PB given in Eq. (36), are the most relevant results of this paper.

Finally, we have discussed a possible quantization method in the q -deformed picture. Based on the standard method consisting of the replacement of the (undeformed) PB for canonically conjugate variables with the (undeformed) commutator for the corresponding quantum operator, we have postulated a similar scheme by replacing the q -PB for quantum conjugate variables with the q -deformed commutator of the corresponding q -deformed quantum operators.

In conclusion we would like to mention the possible applications of the q -deformed classical mechanics that we have developed to the study of some relevant physical phenomenologies and in particular to the framework of thermostatics [23] where it could lead to a generalization of the theory in a manner similar to what the classical Tsallis' thermostatics does with respect to the Boltzmann-Gibbs theory [22].

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